Wave Equation

→ from maxwell's equation

$$\nabla \cdot \vec{D} = \rho_v$$

$$\nabla \cdot \vec{B} = 0.$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

→ $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

taking curl on both sides of equation.

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H}).$$

from vector identity

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right). \quad -(1)$$

$$\nabla \cdot \vec{D} = \rho_v$$

$$\epsilon(\nabla \cdot \vec{E}) = \rho_v$$

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon}.} \quad -(2)$$

using (2) in (1).

$$\nabla\left(\frac{\rho_v}{\epsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial \vec{J}}{\partial t} - \mu\epsilon \frac{\partial \vec{E}}{\partial t}.$$

$$\boxed{\nabla^2 \vec{E} - \mu\epsilon \frac{\partial \vec{E}}{\partial t} = \mu \frac{\partial \vec{J}}{\partial t} + \nabla\left(\frac{\rho_v}{\epsilon}\right)} \quad -(3)$$

→ $$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}.$$

taking curl on both sides of equation.

$$\nabla \times \nabla \times \vec{H} = \nabla \times \vec{J} + \epsilon \frac{\partial}{\partial t}(\nabla \times \vec{E}).$$

from vector identity

$$\nabla(\nabla\cdot\vec{H}) - \nabla^2\vec{H} = (\nabla\times\vec{J}) + \epsilon\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right).$$ ②

$$\nabla\cdot\vec{B} = 0.$$

$$\mu(\nabla\cdot\vec{H}) = 0.$$

$$0 - \nabla^2\vec{H} = (\nabla\times\vec{J}) - \epsilon\frac{\partial^2\vec{B}}{\partial t^2}.$$

$$-\nabla^2\vec{H} = (\nabla\times\vec{J}) - \epsilon\mu\frac{\partial^2 H}{\partial t^2}.$$

$$\boxed{\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2 H}{\partial t^2} = -(\nabla\times\vec{J})} \quad \text{—(4)}.$$

Equation (3) and ④ are plane wave equation.

→ Plane equation for free space.

$\rho_v = 0$, $\vec{J} = 0$.

using these condition in eq (3) & (4).

$$\boxed{\nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0}$$

$$\boxed{\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2 H}{\partial t^2} = 0.}$$

In general,

$$\boxed{\nabla^2\vec{F} - \mu\epsilon\frac{\partial^2 F}{\partial t^2} = 0.}$$

$$V = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} = 3\times10^8 \text{ m/s} = \text{speed of light}$$

→ wave equation for conducting medium | imperfect dielectric / lossy dielectric

$\vec{J} = \sigma\vec{E}$, $\rho_v = 0$.

In eq^n (3) & ④

$$\boxed{\nabla^2\vec{E} - \mu\epsilon\frac{\partial\vec{E}}{\partial t} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = 0}.$$

$$\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2 H}{\partial t^2} = -\sigma(\nabla\times\vec{E}) \Rightarrow \nabla^2\vec{H} - \mu\epsilon\frac{\partial^2 H}{\partial t^2} = +\sigma\frac{\partial\vec{B}}{\partial t}.$$

$$\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2 H}{\partial t^2} - \mu\sigma\frac{\partial\vec{H}}{\partial t} = 0.$$

→ uniform plane waves for free space.

$$\nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0. \quad — (1)$$

$$\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} = 0. \quad — (2)$$

$$\bar{E} = E_0 e^{j\omega t}$$

$$\frac{\partial\bar{E}}{\partial t} = j\omega E_0 e^{j\omega t}.$$

$$\frac{\partial^2\vec{E}}{\partial t^2} = (j\omega)^2 E_0 e^{j\omega t}$$

$$= -\omega^2 E_0 e^{j\omega t}.$$

$$\boxed{\frac{\partial^2\vec{E}}{\partial t^2} = -\omega^2\vec{E}.} \quad — (3) \qquad \boxed{\text{lly } \frac{\partial^2\vec{H}}{\partial t^2} = -\omega^2\vec{H}.} \quad — (4)$$

using (3) in (1).

$$\boxed{\nabla^2\vec{E} + \mu\epsilon\omega^2\frac{\partial^2 E}{\partial t^2} = 0} \quad — (5)$$

using (4) in (2).

$$\boxed{\nabla^2\vec{H} + \omega^2\mu\epsilon\frac{\partial^2 H}{\partial t^2} = 0} \quad — (6)$$

where $\beta = \omega\sqrt{\mu\epsilon}$.

Eq (5) & (6) are called Helmoltz Equation.

→ Since the electric field is in x-direction & is varying w.r.t z axis eq (5) can be written as

$$\frac{\partial^2 E_x}{\partial z^2} + \beta^2\vec{E} = 0.$$

on solving $\boxed{E_x = E_m^+ \cos(\omega t - \beta z) + E_m^- \cos(\omega t + \beta z)}$

from the last two eqns.

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon).$$

$$\boxed{\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}} \quad -(3).$$

using eq(3) in (1) & 2

$$\boxed{\nabla^2 \vec{E} = \gamma^2 \vec{E}}$$

$$\boxed{\nabla^2 \vec{H} = \gamma^2 \vec{H}}.$$

Power Flow and Poynting Theorem.

Statement :- The vector product of electric field intensity ($\vec{E}$) and magnetic field intensity ($\vec{H}$) at any point is a measure of rate of energy flow at that point.

$$\vec{P} = \vec{E} \times \vec{H}$$

where $\vec{P}$ = power flow.
$\vec{E}$ = electric field intensity
$\vec{H}$ = magnetic field intensity.

:- $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{—(1)}.$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{—(2)}.$$

[According to vector identity
$\nabla \cdot (A \times B) = B(\nabla \times \vec{A}) - A(\nabla \times \vec{B})$]

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H}(\nabla \times \vec{E}) - \vec{E}(\nabla \times \vec{H}).$$

on substituting values,

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H}\left(-\frac{\partial \vec{B}}{\partial t}\right) - \vec{E}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right).$$

$$\boxed{\nabla \cdot (\vec{E} \times \vec{H}) = -\vec{H}\frac{\partial \vec{B}}{\partial t} - \vec{E}\cdot\vec{J} - \vec{E}\frac{\partial \vec{D}}{\partial t}} \quad \text{— (1)}$$

Consider,
$$\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) = \vec{H}\cdot\frac{\partial \vec{H}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{H}}{\partial t}.$$

$$\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) = 2\vec{H}\frac{\partial \vec{H}}{\partial t}.$$

$$\frac{\partial}{\partial t}H^2 = 2\vec{H}\cdot\frac{\partial \vec{H}}{\partial t}. \Rightarrow \boxed{\frac{1}{2}\frac{\partial H^2}{\partial t^2} = \vec{H}\cdot\frac{\partial \vec{H}}{\partial t}.}$$

IIIly. $\frac{1}{2}\frac{\partial E^2}{\partial t} = \vec{E}\cdot\frac{\partial \vec{E}}{\partial t}$.

W.K.T. $-\vec{E}\cdot\vec{J} = -\vec{E}(\sigma\vec{E}) = -\sigma E^2$. — (2)

$$-\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} = -\vec{E}\frac{\partial}{\partial t}(\epsilon\vec{E}) = -\epsilon\,\vec{E}\cdot\frac{\partial \vec{E}}{\partial t}$$

$$= -\frac{\epsilon}{2}\frac{\partial E^2}{\partial t^2}. \quad \text{— (3)}$$

$$-\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} = -\vec{H}\frac{\partial}{\partial t}(\mu\vec{H}) = -\mu\,\vec{H}\cdot\frac{\partial \vec{H}}{\partial t}$$

$$= \frac{-\mu}{2}\frac{\partial H^2}{\partial t^2}. \quad \text{— (4)}$$

use (2), (3) & (4) in eq (1)

$$\nabla\cdot(\vec{E}\times\vec{H}) = \frac{-1}{2}\frac{\partial}{\partial t}\left[\mu H^2 + \epsilon E^2\right] - \sigma E^2.$$

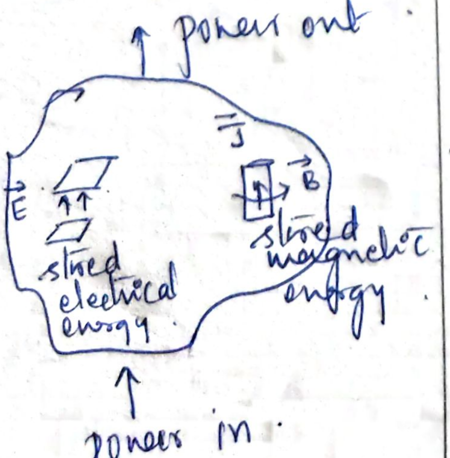
taking volume integral on both sides

$$\int_v \nabla\cdot(\vec{E}\times\vec{H})\,dv = -\int_v \frac{\partial}{\partial t}\left[\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right]dv - \int_v \sigma E^2\,dv.$$

Applying divergence theorem.

$$\cancel{\oint_s (\vec{E}\times\vec{H})\cdot\vec{ds} = -\frac{\partial}{\partial t}\int_v \frac{1}{2}}$$

$$\boxed{\oint_s (\vec{E}\times\vec{H})\cdot\vec{ds} = -\int_v \frac{\partial}{\partial t}\left[\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right]dv - \int_v \sigma E^2\,dv}$$



$\oint_s (\vec{E}\times\vec{H})\,ds$ = rate of energy flow into the volume through its surface.

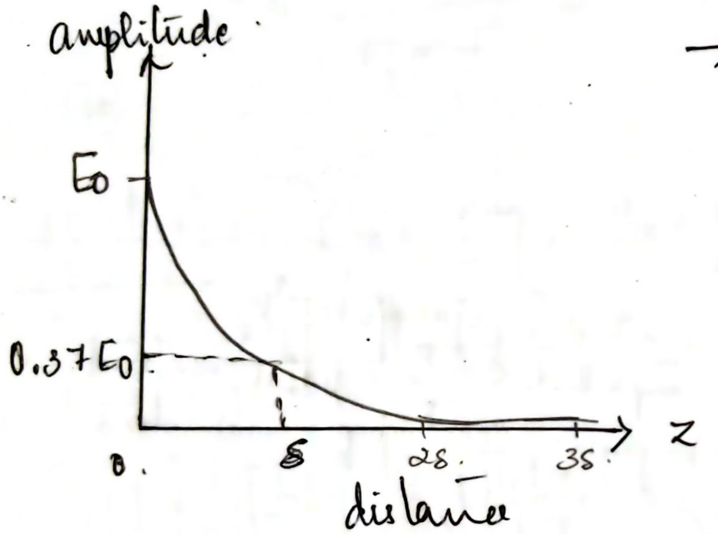
$-\int_v \frac{\partial}{\partial t}\left[\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right]dv$ = rate at which the stored field energy is decreasing in a volume V over which the integration is carried out.

$-\cancel{\oint_s (\vec{E}\times\vec{H})\,ds}$: - ~~rate of~~

$\int_v \vec{E}\cdot\vec{J}\,dv$ = the power dissipated in a volume V.

Skin depth and Skin Effect

When an electromagnetic wave enters a conducting medium its amplitude decreases exponentially and becomes practically zero as shown in fig. after penetrating a small distance. As a result, the current induced by the wave exist only near the surface of the conductor. This effect is called as skin effect.



→ At $z=0$, amplitude of the component E_x is E_m, while at $z=d$, amplitude is $E = E_0 e^{-\alpha d}$.

where α = attenuation constant.

If ~~α~~ $\delta = \frac{1}{\alpha}$.

In distance $z=d$, the amplitude gets reduced by ~~the~~ factor $e^{-\alpha d}$.

→ If $d = \frac{1}{\alpha}$; $E = E_0 e^{-\alpha \cdot \frac{1}{\alpha}} = \frac{E_0}{e}$.

So amplitude decreases to approximately 37% of its original value.

→ Thus skin depth or depth of penetration is defined as the depth of a conductor at which the amplitude of an incident wave drops to 1/e times of its value of amplitude at the time of incidence.

→ For good conductor,

$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$; $\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi f\mu\sigma}} = \frac{1}{\sqrt{\pi f\mu\sigma}}$.

For good conductors, since σ is very high, δ (9)
will be very small, further it depends on
inverse of the frequency f.
Higher frequency lesser will be penetration
eg. for Copper $\sigma = 5.8 \times 10^7$ S/m.

$$\delta = \frac{0.666}{\sqrt{f}}.$$